Graph Theory

Subject: Advanced Discrete Mathematics

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Introduction

Graph theory is a branch of discrete mathematics that studies graphs, which consist of **vertices (nodes)** and **edges (connections)**. It was first introduced by **Euler** in 1736 through the **Königsberg Bridge Problem**. Graphs can be **directed or undirected**, **weighted or unweighted**, and have various applications in **computer science, network design, social networks, transportation, and biology**. Graph theory plays a crucial role in solving real-world problems related to optimization, data organization, and communication networks.

Definition of Graph

- A graph is formally defined as:

$$G=(V,E)$$

where **V** is a set of vertices and **E** is a set of edges connecting these vertices.

Types of Graph

Directed Graph: A directed graph is a graph in which each element e of E(G) is assigned an ordered pair of vertices (a,b) along with arrow starting from a to b, where a is called initial point and b is called terminal point.

Undirected graph: A graph where edges have no direction.

Simple Graph: A graph which has neither loop nor parallel edges is called simple graph.

General Graph: A graph which has either loop or parallel edges is called simple graph.

Complete Graph: A simple graph in which there exists an edge between every pair of vertices is called complete graph.

Subgraphs

Let G and H be two graphs with vertex sets V(H), V(G) and edge sets E(H),E(G) such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is said to be a subgraph of G.

Example: If G = (V, E), then a subgraph G' = (V', E') where $V' \subseteq V$ and $E' \subseteq E$.

Walks, Paths and circuits

Open Walk: If a walk begin and end with the different vertices, it is called open walk.

Closed Walk: If the initial and terminal vertices of a walk are same, it is called closed walk.

Path: An open walk in which no vertex appear more than once is called path.

Circuit: A circuit is a closed walk in which no vertex appear more than once.

Eulerian, Hamiltonian Graphs and Planar Graphs

Eulerian Graph: A graph containing a Eulerian circuit (visiting every edge exactly once).

Hamiltonian Graph: A graph containing a Hamiltonian cycle (visiting every vertex exactly once).

Planar Graph: A planar graph can be drawn without edges crossing each other.

- Example: The famous Kuratowski's theorem identifies non-planar graphs.

Some Important Theorems

Handshaking Theorem: The sum the degrees of all the vertices in a graph G is equal to the twice the number of edges.

Euler's Formula: Let G=(V,E) be a connected planar graph and let R be the number of regions defined by any planar depiction of G. Then R=|E|-|V|+2.

Applications of Graph Theory

- Graph Theory is used in:

1. Computer Networks (Routing)
2. Social Networks (Connections)
3. Transportation Planning
4. Biology (Gene Networks)

Conclusion

- Graph theory provides powerful tools for solving complex problems in various fields, from computer science to logistics.